A proof of the existence of indiscernible trees without Erdos-Rado theorem (Model theoretic aspects of the notion of independence and dimension)

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A proof of the existence of indiscernible trees without Erdös-Rado theorem

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Our interest in this paper is to see the similarity between Erdös-Rado theorem and compactness argument using Ramsey theorem in model theory. Erdös-Rado theorem is a theorem in infinitary combinatorics that generalizes Ramsey theorem to handle uncountable situations. In model theory, compactness arguments are available, so arguments tend to be settled in countable situation.

We give a proof without Erdös-Rado theorem to the next theorem.

Theorem 3.1.16. *Let B be a set of parameters, and $\Gamma(x_{\omega^{<\omega}})$ be a set of $\mathcal{L}_B$-formulas. If $\Gamma(x_{\omega^{<\omega}})$ has $\mathcal{L}_{\mathrm{S}}$-subtree property, then Γ is realized by an $\mathcal{L}_{\mathrm{S}}$-indiscernible tree over B.*

This theorem is proved with Erdös-Rado theorem in [2] and [3], while we use compactness arguments and Ramsey theorem.

Byunghan Kim, Hyeung-Joon Kim, and Lynn Scow recently revised their preprint[4], and it contains essentially the same argument of this paper. We have constructed the content independently.

We work in a complete theory T in a language $\mathcal{L}$ throughout this paper. Let $\mathbb{M}$ be a big model of T. We write $\langle n_1 \dots n_k\rangle$ to refer the element of $\omega^{<\omega}$ of length k whose i-th value is n_i. For $\eta_1, \eta_2 \in \omega^{<\omega}$, we write $\eta_1 \frown \eta_2$ to refer the concatenation of η_1 and η_2. For a set S and an indexed set $(a_s)_{s\in S}$, we write a_S to denote $(a_s)_{s\in S}$.

1 Theorems in infinitary combinatorics

1.1 Ramsey's theorem and Erdös-Rado theorem

Infinite Ramsey's theorem and Erdös-Rado theorem are theorems in infinitary combinatorics. Erdös-Rado theorem is a generalization of Ramsey's theorem to uncountable situations.

Definition 1.1.1. For cardinals α, β, γ and for $n < \omega$, we write

$$\alpha \to (\beta)^n_\gamma$$

whenever $|X| = \alpha$ and $f : [X]^n \to \gamma$, there exists $Y \subset X$ with $|Y| = \beta$ such that $f([Y]^n)$ is a singleton.

Theorem 1.1.2 (Infinite Ramsey's Theorem). *For all $k, n \in \omega$,*

$$\aleph_0 \to (\aleph_0)^n_k.$$

Theorem 1.1.3 (Erdös-Rado Theorem). *For all $n \in \omega$ and infinite cardinal κ,*

$$\exp_n(\kappa)^+ \to (\kappa^+)^{n+1}_\kappa,$$

where $\exp_n(\kappa)$ *is inductively defined by* $\exp_0(\kappa) = \kappa$, $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$.

2 Indiscernible structures

We introduce indiscernible sequences and $\mathcal{L}_S/\mathcal{L}_1$-indiscernible trees. We also define subsequence property and $\mathcal{L}_S/\mathcal{L}_1$-subtree property, which later we prove that they induces the existence of indiscernible structures.

2.1 Indiscernible sequences

Definition 2.1.4 (Indiscernible sequences). Let $\mathcal{L}_o = \{<\}$ and $\mathcal{L}_o$-structure I be a totally ordered set, and let $B \subset \mathbb{M}$. For $a_I \subset \mathbb{M}$, we say a_I is an indiscernible sequence over B if for all $I_0, I_1 \subset I$ such that $I_0 \underset{\mathcal{L}_o}{\simeq} I_1$, it holds that $\mathrm{tp}(a_{I_0}/B) = \mathrm{tp}(a_{I_1}/B)$.

Be careful the index set I is not a subset of the big model $\mathbb{M}$ and the I-indexed set a_I is a subset of $\mathbb{M}$.

Subsequence property was introduced by Tsuboi in his lecture note in 1999.

Definition 2.1.5 (Subsequence property). Let $\mathcal{L}_o = \{<\}$ and $\mathcal{L}_o$-structure I be a totally ordered set. For a set of formulas $\Gamma(x_I)$, we say Γ has subsequence property if

$$\cup\{\, \Gamma(x_{\sigma(I)}) \mid \sigma : I \to I \text{ is an } \mathcal{L}_o\text{-embedding} \,\}$$

is consistent.

Example 2.1.6. *Let $\Gamma(x_\omega)$ be the set of formulas expressing "x_ω is an indiscernible sequence." Then, Γ has subsequence property. Γ can be concretely written as*

$$\left\{ \varphi(x_I) \leftrightarrow \varphi(x_J) \,\middle|\, \varphi \in \mathcal{L},\ I, J \subset \omega,\ I \underset{\mathcal{L}_o}{\simeq} J \right\}.$$

Example 2.1.7. *Let $\Gamma(x_\omega, y_\omega)$ be the set of formulas expressing "$(x_i, y_i)_{i \in \omega}$ witnesses the order property of $\varphi(x, y)$." Then, Γ has the subsequence property.*

Γ can be concretely written as

$$\{\, \varphi(x_i, y_j) \mid i < j < \omega \,\} \cup \{\, \neg\varphi(x_j, y_i) \mid j \le i < \omega \,\}.$$

The following lemma guarantees the existence of indiscernible sequences.

Lemma 2.1.8 (Tsuboi 1999). *Let B be a set of parameters, and $\Gamma(x_\omega)$ be a set of $\mathcal{L}_B$-formulas. If $\Gamma(x_\omega)$ has subsequence property, then Γ is realized by an indiscernible sequence over B.*

Proof. We show $\Gamma(x_\omega) \cup$ "x_ω is an indiscernible sequence over B" is consistent, where

"x_ω is an indiscernible sequence over B" =

$$\left\{ \varphi(x_I) \leftrightarrow \varphi(x_J) \,\middle|\, \varphi \in \mathcal{L}_B,\ I, J \underset{\text{fin}}{\subset} \omega,\ I \underset{\mathcal{L}_o}{\simeq} J \right\}.$$

We use compactness argument. We fix $\mathcal{L}_B$-formulas $\varphi_1, \dots, \varphi_m$ each of which has n free variables from x_I. It is sufficient to show

$$\widetilde{\Gamma} = \Gamma \cup \left\{ \varphi_k(x_{I_0}) \leftrightarrow \varphi_k(x_{J_0}) \,\middle|\, k = 1, \dots, m,\ I_0, I_1 \underset{n \text{ elem}}{\subset} \omega,\ I_0 \underset{\mathcal{L}_o}{\simeq} I_1 \right\}$$

is consistent. We fix a realization $A \models \Gamma$, and we define $F : A^n \to 2^n$ by

$$F(\bar{a}) = \textstyle\sum_{k=1}^{n} i_k 2^k, \text{ where } \begin{cases} i_k = 0 & \text{if } \neg\varphi_k(\bar{a}) \text{ holds} \\ i_k = 1 & \text{if } \varphi_k(\bar{a}) \text{ holds.} \end{cases} \qquad \text{for } \bar{a} \in A^n$$

By Ramsey's theorem, there is an infinite $A' \subset A$ such that $F|_{A'^n}$ is constant. This A' is a witness of $\widetilde{\Gamma}$, for φ_k have the same truth value on A'^n, and $A' \models \Gamma$ by subsequence property. □

2.2 Indiscernible trees

Definition 2.2.9. Let $\mathcal{L}_1 = \{ \cap, <_{\text{len}}, <_{\text{lex}}, <_{\text{ini}} \}$, and let $\mathcal{L}_S = \mathcal{L}_1 \cup \{ P_n \mid n \in \omega \}$.

Here, we use the notation $\mathcal{L}_S$ instead of the original notation $\mathcal{L}_0$ in [2].

Definition 2.2.10. Let the interpretation of $\mathcal{L}_1$ and $\mathcal{L}_S$ in $\omega^{<\omega}$ as follows:

- $\eta \cap \nu =$ the longest common initial segment of η and ν.
- $\eta <_{\text{len}} \nu \Leftrightarrow \eta$ has the less length than ν.
- $\eta <_{\text{lex}} \nu \Leftrightarrow \eta$ is less than ν in the lexicographic order.
- $\eta <_{\text{ini}} \nu \Leftrightarrow \eta$ is a proper initial segment of ν.
- $P_n(\eta) \Leftrightarrow \eta$ has the length of n.

We refer $\mathcal{L}_S$ or $\mathcal{L}_1$-substructures of $\omega^{<\omega}$ by the word 'trees'.

Definition 2.2.11 (Indiscernible trees). Let $B \subset \mathbb{M}$.

(1) Let S be an $\mathcal{L}_S$-substructure of $\omega^{<\omega}$. For $a_S \subset \mathbb{M}$, we say a_S is an $\mathcal{L}_S$-indiscernible tree over B if for all $S_0, S_1 \subset S$ such that $S_0 \underset{\mathcal{L}_S}{\simeq} S_1$, it holds that $\text{tp}(a_{S_0}/B) = \text{tp}(a_{S_1}/B)$.

(2) Let S be an $\mathcal{L}_1$-substructure of $\omega^{<\omega}$. For $a_S \subset \mathbb{M}$, we say a_S is an $\mathcal{L}_1$-indiscernible tree over B if for all $S_0, S_1 \subset S$ such that $S_0 \underset{\mathcal{L}_1}{\simeq} S_1$, it holds that $\text{tp}(a_{S_0}/B) = \text{tp}(a_{S_1}/B)$.

Be careful the index set S is not a subset of the big model $\mathbb{M}$ and the S-indexed set a_S is a subset of $\mathbb{M}$.

Example 2.2.12. *For $\eta, \nu \in \omega^{<\omega}$, we say η is an ancestor or a descendant of ν if either of the nodes is an proper initial segment of the other, and we say η and ν are siblings if η and ν has the same length n and the length of $\eta \cap \nu$ is $n-1$.*

Let T be the theory of random graph in the language $\{ R(*,*) \}$. For distinct vertices $a_{\omega^{<\omega}}$ in the big model that satisfies for all $\eta, \nu \in \omega^{<\omega}$

$$\models R(a_\eta, a_\nu) \ \Leftrightarrow\ \text{“}\eta \textit{ is an ancestor or a descendant of } \nu\text{”}$$

form an $\mathcal{L}_S$ and $\mathcal{L}_1$-indiscernible tree.

Let $b_{\omega^{<\omega}}$ be the tree-indexed subset such that for all $\eta, \nu \in \omega^{<\omega}$,

$$\models R(a_\eta, a_\nu) \Leftrightarrow \text{“}\eta \text{ is an ancestor or a descendant of } \nu\text{” or “}\eta \text{ and } \nu \text{ are siblings.”}$$

Then, b_ω is an $\mathcal{L}_S$-indiscernible tree but not an $\mathcal{L}_1$-indiscernible tree. In fact,

$$\{\emptyset, \langle 0\rangle, \langle 1\rangle\} \underset{\mathcal{L}_1}{\simeq} \{\emptyset, \langle 00\rangle, \langle 10\rangle\} \text{ but } \models R(b_{\langle 0\rangle}, b_{\langle 1\rangle}) \wedge \neg R(b_{\langle 00\rangle}, b_{\langle 10\rangle}).$$

Definition 2.2.13 (Subtree property [2], [3]). Let $B \subset \mathbb{M}$.

(1) Let S be an $\mathcal{L}_S$-substructure of $\omega^{<\omega}$. For a set of $\mathcal{L}_B$-formulas $\Gamma(x_S)$, we say Γ has $\mathcal{L}_S$-subtree property if
$$\cup\{\, \Gamma(x_{\sigma(S)}) \mid \sigma : I \to I \text{ is an } \mathcal{L}_S\text{-embedding} \,\}$$
is consistent.

(2) Let S be an $\mathcal{L}_1$-substructure of $\omega^{<\omega}$. For a set of $\mathcal{L}_B$-formulas $\Gamma(x_S)$, we say Γ has $\mathcal{L}_S$-subtree property if
$$\cup\{\, \Gamma(x_{\sigma(S)}) \mid \sigma : I \to I \text{ is an } \mathcal{L}_1\text{-embedding} \,\}$$
is consistent.

Example 2.2.14. $\Gamma(x_{\omega^{<\omega}}) =$ *“$x_{\omega^{<\omega}}$ witnesses the k-tree property of $\varphi(x, y)$” has the $\mathcal{L}_S$-subtree property (if Γ is consistent).*

Γ can be concretely written as

$$\Gamma(y_{\omega\times\omega}) = \bigcup_{i\in\omega}\left\{ \neg\exists x \left(\bigwedge_{l<k} \varphi(x, y_{i^\frown j_l}) \right) \,\middle|\, j_0, \ldots, j_{k-1} \in \omega \right\} \cup \bigcup_{\nu\in\omega^\omega} \left\{ \exists x \left(\bigwedge_{i<n} \varphi(x, y_{\nu|_i}) \right) \,\middle|\, n \in \omega \right\}.$$

3 Existence of indiscernible trees

In this section, we prove that subtree property implies the existence of an indiscernible tree without Erdös-Rado theorem.

The existence of $\mathcal{L}_S$-indiscernible trees is proved with the following theorem in [2], [3].

Theorem (Shelah, Theorem 2.6 of [5, p.662]). For all $k, n \in \omega$ and ordinal μ, there exists an ordinal λ such that for any $f : (\lambda^{<n})^k \to \mu$, there is an $\mathcal{L}_S$-substructure $S \subset \lambda^{<n}$ with $S \underset{\mathcal{L}_S}{\simeq} \omega^{<\omega}$ satisfying $f(X) = f(Y)$ for all $X, Y \in S^k$ with $X \underset{\mathcal{L}_S}{\simeq} Y$

This is a variation of Erdös-Rado theorem regarding trees. We want to show the existence of indiscernible trees without this theorem.

3.1 $\mathcal{L}_S$-indiscernible trees

Proposition 3.1.15 ([3]). *Let B be a set of parameters, and $\Gamma(x_\omega^{<n})$ be a set of $\mathcal{L}_B$-formulas for $n \in \omega$. If $\Gamma(x_{\omega^{<n}})$ has the $\mathcal{L}_S$-subtree property, then Γ is realized by an $\mathcal{L}_S$-indiscernible tree over B.*

Proof. We show $\Gamma(x_{\omega^{<n}}) \cup$ "$x_{\omega^{<n}}$ is an $\mathcal{L}_{\mathrm{S}}$-indiscernible tree over B" is consistent, where

"$x_{\omega^{<n}}$ is an $\mathcal{L}_{\mathrm{S}}$-indiscernible tree over B" =

$$\left\{ \varphi(x_S) \leftrightarrow \varphi(x_T) \;\middle|\; \varphi \in \mathcal{L}_B,\ S,T \underset{\text{fin}}{\subset} \omega^{<n},\ S \underset{\mathcal{L}_{\mathrm{S}}}{\simeq} T \right\}.$$

We show this by induction on n. The case $n = 1$ is clear because $\omega^{<1} = \{\emptyset\}$.

Suppose the n case holds. We write $k^\frown\omega^{<n}$ to denote the set $\{\sigma \in \omega^{<n+1} \mid \sigma(0) = k\}$ and X_k to denote the set of variables $x_{k^\frown\omega^{<n}}$.

Claim A. $\Gamma(x_{\omega^{<n+1}}) \cup \left(\bigcup_{k \in \omega} \Sigma_k(x_{\omega^{<n+1}})\right)$ *is consistent, where*

$$\begin{aligned}\Sigma_k &= \text{“}X_k \textit{ is an } \mathcal{L}_{\mathrm{S}}\textit{-indiscernible tree over } Bx_\emptyset X_0X_1\dots X_{k-1}X_{k+1}\dots\text{”} \\ &= \left\{ \varphi(x_S) \leftrightarrow \varphi(x_T) \;\middle|\; \begin{array}{l}\varphi \in \mathcal{L}(Bx_\emptyset X_0X_1\dots X_{k-1}X_{k+1}\dots), \\ S,T \underset{\text{fin}}{\subset} k^\frown\omega^{<n},\ S \underset{\mathcal{L}_{\mathrm{S}}}{\simeq} T\end{array} \right\}.\end{aligned}$$

Proof of Claim A. Let $a_{\omega^{<n+1}} = a_\emptyset A_0A_1\dots \models \Gamma$, where $A_k = a_{k^\frown\omega^{<n}}$. First, observe that for any tree S with $S \underset{\mathcal{L}_{\mathrm{S}}}{\simeq} \omega^{<n}$, the tree $\emptyset\ \ 0^\frown S\ \ 1^\frown\omega^{<n}\ \ 2^\frown\omega^{<n}\dots$ becomes an $\mathcal{L}_{\mathrm{S}}$-substructure that is isomorphic to whole $\omega^{<n+1}$. Therefore $\Gamma(a_\emptyset X_0A_1A_2\dots)$ has $\mathcal{L}_{\mathrm{S}}$-subtree property over $a_\emptyset A_1A_2\dots$ by the $\mathcal{L}_{\mathrm{S}}$-subtree property of $\Gamma(x_{\omega^{<n}})$. By induction hypothesis, $\Gamma(a_\emptyset X_0A_1\dots)$ is realized by A_0' which is an $\mathcal{L}_{\mathrm{S}}$-indiscernible tree over $a_\emptyset A_1A_2\dots$, *i.e.* $\Gamma \cup \Sigma_0$ is consistent.

Similarly, $(\Gamma\cup\Sigma_0)(a_\emptyset A_0'X_1A_2\dots)$ has subtree property over $a_\emptyset A_0'A_2\dots$. Again by induction hypothesis $(\Gamma\cup\Sigma_0)(a_\emptyset A_0'X_1A_2\dots)$ is realized by A_1', an $\mathcal{L}_{\mathrm{S}}$-indiscernible tree over $a_\emptyset A_0'A_2\dots$. Notice A_0' is still an $\mathcal{L}_{\mathrm{S}}$-indiscernible tree over $a_\emptyset A_1'A_2\dots$, since especially $\Sigma_0(a_\emptyset A_0'A_1'A_2\dots)$ holds. Hence, $\Gamma\cup\Sigma_0\cup\Sigma_1$ is consistent.

Iterating this procedure m times, $\Gamma(x_{\omega^{<n+1}}) \cup \left(\bigcup_{k=0}^{m-1} \Sigma_k(x_{\omega^{<n+1}})\right)$ is consistent. By compactness, we have shown the claim. $\boxed{\text{end of the proof of Claim A}}$

Let $\Gamma'(x_{\omega^{<n+1}}) = \Gamma(x_{\omega^{<n+1}}) \cup \left(\bigcup_{k\in\omega} \Sigma_k(x_{\omega^{<n+1}})\right)$.

Claim B. $\Gamma'(x_{\omega^{<n+1}}) \cup$ "$X_0X_1\dots$ *is an indiscernible sequence over* $Bx_\emptyset$" *is consistent, where*

$$\begin{aligned}&\text{“}X_0X_1\dots \textit{ indiscernible sequence over } Bx_\emptyset\text{”} \\ &\quad = \{\varphi(X_{i_0},\dots,X_{i_m}) \leftrightarrow \varphi(X_{j_0},\dots,X_{j_m}) \mid \varphi\in\mathcal{L}_{Bx_\emptyset},\ i_0<\dots<i_m,\ j_0<\dots<j_m\}.\end{aligned}$$

Proof of Claim B. First, observe that for any subsequence $(i_k{}^\frown\omega^{<n})_{k\in\omega}$ of $(i^\frown\omega^{<n})_{i\in\omega}$, the tree $x_\emptyset\ i_0{}^\frown\omega^{<n}\ i_1{}^\frown\omega^{<n}\ i_2{}^\frown\omega^{<n}\dots$ is $\mathcal{L}_{\mathrm{S}}$-isomorphic to the whole $x_{\omega^{<n+1}}$. Since $\Gamma'(x_{\omega^{<n+1}})$ has subtree property over B, $\Gamma'(x_\emptyset X_0X_1\dots)$ has subsequence property over $Bx_\emptyset$. Therefore, there is a realization $a_{\omega^{<n+1}} = a_\emptyset A_0A_1\dots$ of Γ', where $A_k = a_{k^\frown\omega^{<n}}$, such that $A_0A_1\dots$ is an indiscernible sequence over $Ba_\emptyset$. This can be shown by an argument similar to the proof of Lemma 2.1.8. $\boxed{\text{end of the proof of Claim B}}$

Let $\Gamma''(x_{\omega^{<n+1}}) = \Gamma'(x_{\omega^{<n+1}}) \cup$ "$X_0X_1\dots$ is an indiscernible sequence over $Bx_\emptyset$".

Claim C. *A realization of* $\Gamma''(x_{\omega^{<n+1}})$ *is an* $\mathcal{L}_{\mathrm{S}}$*-indiscernible tree realizing* Γ.

Proof of Claim C. Let $\varphi \in \mathcal{L}_B$, $S, T \underset{\text{fin}}{\subset} \omega^{<n+1}$ such that $S \underset{\mathcal{L}_S}{\simeq} T$, and $\theta \equiv \varphi(x_S) \leftrightarrow \varphi(x_T)$. We show $\Gamma'' \vdash \theta$. S, T have the form of

$$S = \bigcup_{k=1}^{m} S_{i_k}, \quad S_{i_k} = \{ \nu \in S \mid \nu(0) = i_k \},\ i_0 < \cdots < i_m,$$
$$T = \bigcup_{k=1}^{m} T_{j_k}, \quad T_{j_k} = \{ \nu \in T \mid \nu(0) = j_k \},\ j_0 < \cdots < j_m.$$

Let $\sigma : \bigcup_{k=1}^{m} i_k{}^\frown\omega^{<n} \to \bigcup_{k=1}^{m} j_k{}^\frown\omega^{<n}$ be the natural isomorphism. Since $\Gamma''(x_{\omega^{<n+1}}) \supset$ "$X_0X_1\dots$ is an indiscernible sequence over $Bx_\emptyset$",

$$\Gamma''(x_{\omega^{<n+1}}) \vdash \varphi(x_\emptyset x_{S_{i_0}} \dots x_{S_{i_m}}) \leftrightarrow \varphi(x_\emptyset x_{\sigma(S_{i_0})} \dots x_{\sigma(S_{i_m})}).$$

We have $S \underset{\mathcal{L}_S}{\simeq} T$ and so $\sigma(S_{i_k}) \underset{\mathcal{L}_S}{\simeq} T_{j_k}$ for each $k = 1, \dots, m$. Since $\Gamma''(x_{\omega^{<n+1}}) \supset$ "X_k is an $\mathcal{L}_S$-indiscernible tree over $Bx_\emptyset X_0X_1 \dots X_{k-1}X_{k+1}\dots$" for all $k \in \omega$, it holds that

$$\Gamma''(x_{\omega^{<n+1}}) \vdash \varphi(x_\emptyset x_{\sigma(S_{i_0})} \dots x_{\sigma(S_{i_m})}) \leftrightarrow \varphi(x_\emptyset x_{T_{j_0}} \dots x_{T_{j_m}}).$$

Thus we have shown $\Gamma''(x_{\omega^{<n+1}}) \vdash \theta$. $\boxed{\text{end of the proof of Claim C}}$

From the above argument, we have shown the $n+1$ case of proposition. □

Theorem 3.1.16 ([3]). *Let B be a set of parameters, and $\Gamma(x_{\omega^{<\omega}})$ be a set of $\mathcal{L}_B$-formulas. If $\Gamma(x_{\omega^{<\omega}})$ has the $\mathcal{L}_S$-subtree property, then Γ is realized by an $\mathcal{L}_S$-indiscernible tree over B.*

Proof. This is an immediate consequence from Proposition 3.1.15 and Compactness. □

Example 3.1.17. $\Gamma(x_{\omega^{<\omega}}) =$ "*$x_{\omega^{<\omega}}$ witnesses the k-tree property of $\varphi(x, y)$*" *is realized by an $\mathcal{L}_S$-indiscernible tree (if Γ is consistent).*

3.2 $\mathcal{L}_1$-indiscernible trees

Definition 3.2.18 ([3]). Let X be a substructure of $\omega^{<\omega}$, *i.e.* X is closed under the binary function $\cap$. We define level(X) by level$(X) = \{ \operatorname{dom}(\eta) \mid \eta \in X \}$.

Lemma 3.2.19 ([3]). *Let $n \in \omega$ and X, Y be n-element substructures of $\omega^{<\omega}$. $X \underset{\mathcal{L}_S}{\simeq} Y$ if and only if $X \underset{\mathcal{L}_1}{\simeq} Y$ and* level$(X) =$ level(Y).

Proof. If we have $X \underset{\mathcal{L}_S}{\simeq} Y$, then $X \underset{\mathcal{L}_1}{\simeq} Y$ and level$(X) =$ level(Y) clearly holds.

Suppose $X \underset{\mathcal{L}_1}{\simeq} Y$ and level$(X) =$ level(Y) holds. We put $l = |\text{level}(X)| = |\text{level}(Y)|$ and fix the $\mathcal{L}_1$-isomorphism $\sigma : X \to Y$. Let $(\eta_i)_{i<n}$ enumerates X and $\nu_i = \sigma(\eta_i)$ for $i < n$. There are $i_1, \dots, i_l$ such that $\eta_{i_1} <_{\text{ini}} \cdots <_{\text{ini}} \eta_{i_l}$ and so $\nu_{i_1} <_{\text{ini}} \cdots <_{\text{ini}} \nu_{i_l}$. By the condition level$(X) =$ level(Y), we have $\operatorname{dom}(\eta_{i_k}) = \operatorname{dom}(\nu_{i_k})$ for each $1 \le k \le l$. Since $\mathcal{L}_1$-isomorphisms do not change the relation of having the same length, we have $\operatorname{dom}(\eta) = \operatorname{dom}(\sigma(\eta))$ thus $P_m(\eta) \leftrightarrow P_m(\sigma(\eta))$ for all $\eta \in X$ and $m \in \omega$. Hence σ is the $\mathcal{L}_S$-isomorphism between X and Y. □

Theorem 3.2.20 ([3]). *Let B be a set of parameters, and $\Gamma(x_{\omega^{<\omega}})$ be a set of $\mathcal{L}_B$-formulas. If $\Gamma(x_{\omega^{<\omega}})$ has the $\mathcal{L}_1$-subtree property, then Γ is realized by an $\mathcal{L}_1$-indiscernible tree over B.*

Proof. We show the set of $\mathcal{L}_B$-formulas

$$\bar{\Gamma}(x_{\omega^{<\omega}}) = \Gamma \cup \left\{ \varphi(x_{X_1}) \leftrightarrow \varphi(x_{X_2}) \;\middle|\; \begin{array}{l} \varphi \text{ is an } \mathcal{L}_B\text{-formula,} \\ X_1, X_2 \text{ are finite subsets of } \omega^{<\omega} \text{ with } X_1 \underset{\mathcal{L}_1}{\simeq} X_2 \end{array} \right\}$$

is consistent.

Claim. *For a finite substructure X of $\omega^{<\omega}$ and an $\mathcal{L}_B$-formula $\varphi(x_X)$,*

$$\Gamma_\varphi(x_{\omega^{<\omega}}) = \Gamma \cup \left\{ \varphi(x_{X_1}) \leftrightarrow \varphi(x_{X_2}) \;\middle|\; X_1, X_2 \text{ are subsets of } \omega^{<\omega} \text{ with } X_1 \underset{\mathcal{L}_1}{\simeq} X_2 \underset{\mathcal{L}_1}{\simeq} X \right\}$$

is consistent.

Proof of Claim. We put $k = |\operatorname{level}(X)|$. Γ has $\mathcal{L}_1$-subtree property so $\mathcal{L}_S$-subtree property. By Proposition 3.1.16, Γ has a realization $a_{\omega^{<\omega}}$ that is an $\mathcal{L}_S$-indiscernible tree over B. We define the function $f : [\omega]^k \to \{0, 1\}$ by

$$f(\{n_1, \dots, n_k\}) = \begin{cases} 1 & \text{if } \varphi(a_Y) \text{ holds for all } Y \underset{\mathcal{L}_1}{\simeq} X \text{ with } \operatorname{level}(Y) = \{n_1, \dots, n_k\} \\ 0 & \text{if } \neg\varphi(a_Y) \text{ holds for all } Y \underset{\mathcal{L}_1}{\simeq} X \text{ with } \operatorname{level}(Y) = \{n_1, \dots, n_k\}. \end{cases}$$

This is well defined because $X \underset{\mathcal{L}_1}{\simeq} Y$ and $\operatorname{level}(X) = \operatorname{level}(Y)$ imply $X \underset{\mathcal{L}_S}{\simeq} Y$ and $a_{\omega^{<\omega}}$ is an $\mathcal{L}_S$-indiscernible tree over B. By Ramsey's theorem, there is an infinite $H \subset \omega$ such that f is constant on $[H]^k$. Let h_ω enumerate the elements of H in increasing order. For $\eta \in \omega^{<\omega}$ we define $\sigma_H : \omega^{<\omega} \to \omega^{<\omega}$ by $\operatorname{dom}(\sigma_H(\eta)) = h_{\operatorname{dom}(\eta)}$ and

$$\sigma_H(\eta)(n) = \begin{cases} 0 & \text{if } n \notin H \\ \eta(i) & \text{if } n = h_i \end{cases}$$

$$\text{i.e. } \sigma_H(\eta) = \left\langle \underbrace{0\dots0}_{h_0}\ \eta(0)\ \underbrace{0\dots0}_{h_1 - h_0 - 1}\ \eta(1)\ \underbrace{0\dots0}_{h_2 - h_1 - 1}\ \dots\ \eta(d-1)\ \underbrace{0\dots0}_{h_{d-1} - h_{d-2} - 1} \right\rangle, \text{ where } d = \operatorname{dom}(\eta).$$

Observe that for $\eta, \mu, \nu \in \omega^{<\omega}$ if $\eta <_{\text{len}} \nu$, $\eta <_{\text{ini}} \nu$, $\eta <_{\text{lex}} \nu$, $\eta \cap \nu = \mu$ holds, then we have $\sigma_H(\eta) <_{\text{len}} \sigma_H(\nu)$, $\sigma_H(\eta) <_{\text{ini}} \sigma_H(\nu)$, $\sigma_H(\eta) <_{\text{lex}} \sigma_H(\nu)$, $\sigma_H(\eta) \cap \sigma_H(\nu) = \sigma_H(\mu)$ respectively. Thus σ_H is an $\mathcal{L}_1$-embedding.

By the $\mathcal{L}_1$-indiscernibility of Γ, $(a_{\sigma_H(\eta)})_{\eta \in \omega^{<\omega}}$ is also a realization of Γ, and by the choice of H, $(a_{\sigma_H(\eta)})_{\eta \in \omega^{<\omega}}$ satisfies Γ_φ. Hence Γ_φ is consistent. $\boxed{\text{end of the proof of Claim}}$

Since for any $\mathcal{L}_B$-formula φ and $X \subset \omega^{<\omega}$, Γ_φ in the above claim also has the $\mathcal{L}_1$-subtree property, we can show the finite satisfiability of $\bar{\Gamma}$ using the claim iteratively. $\square$

Example 3.2.21. *Let T be* NTP$_2$ *theory. If $\varphi(x, y)$ has the k-tree property, then there exists $k' \in \omega$ such that the set of formulas $\Gamma_{k'}(x_{\omega^{<\omega}})$ = "$x_{\omega^{<\omega}}$ witnesses the k'-tree property of $\varphi(x, y)$" has the $\mathcal{L}_1$-subtree property, hence $\Gamma_{k'}$ is realized by an $\mathcal{L}_1$-indiscernible tree.*

Here, we give a proof for this example.

Proof. Since the theory is NTP$_2$, there is $l \in \omega$ that satisfies the following condition: for all array of parameters $c_{l \times \omega}$, if $\{\varphi(x, c_{i,j}) \mid j \in \omega\}$ is k-inconsistent for all $i < l$, then there exists $\nu \in \omega^l$ such that $\{\varphi(x, c_{i,\nu(i)}) \mid i < l\}$ is inconsistent. Let k' be $k \times l$, and for $N \in \omega$, let $\Gamma_N(y_{\omega^{<\omega}})$ be the set of formulas "$y_{\omega^{<\omega}}$ witnesses the N-tree property of $\varphi(x, y)$"

Claim. *$\Gamma_{k'}$ has the $\mathcal{L}_1$-subtree property.*

Proof of Claim. We confirm the consistency of $\cup\{\, \Gamma_{k'}(x_{\sigma(I)}) \mid \sigma : I \to I \text{ is an } \mathcal{L}_1\text{-embedding} \,\}$. Since Γ_k has the $\mathcal{L}_S$-subtree property, we can apply Theorem 3.1.16 to obtain an $\mathcal{L}_S$-indiscernible tree $b_{\omega^{<\omega}}$ which realizes Γ_k. Clearly, $b_{\omega^{<\omega}}$ also realizes $\Gamma_{k'}$. We show $b_{\omega^{<\omega}}$ is a realization of $\Gamma_{k'}(y_{\sigma(\omega^{<\omega})})$ for all $\mathcal{L}_1$-embedding σ. The condition "$\{\, \varphi(x, b_{\sigma(\nu|_n)}) \mid n \in \omega \,\}$ is consistent for all $\nu \in \omega^\omega$" clearly holds because an $\mathcal{L}_1$-embedding sends a path into a path and $b_{\omega^{<\omega}}$ is a witness of the k-tree property of φ.

For the condition "$\left\{ \varphi(x, b_{\sigma(\eta^\frown n)}) \,\middle|\, n \in \omega \right\}$ is k'-inconsistent for all $\eta \in \omega^{<\omega}$," since an $\mathcal{L}_1$-embedding preserves the relation of having the same length, it suffices to show any subset $A \subset \omega^{<\omega}$ of k' elements taht have the same length, $\{\, \varphi(x, b_\eta) \mid \eta \in A \,\}$ is inconsistent. Let A be a subset of k' elements in $\omega^{<\omega}$ each of which element has the same length, then either the case happens:

(1) There is k-element subset $A_1 \subset A$ that belongs to the same sequence of siblings.

(2) There is l-element subset $A_2 \subset A$ whose parents are pairwise distinct.

In the case (1), $\{\, \varphi(x, b_\eta) \mid \eta \in A_1 \,\}$ is inconsistent, since all elements in A_1 are contained in a particular sequence of siblings and $b_{\omega^{<\omega}}$ is a witness of the k-tree property of φ.

In the case (2), we put $A_2 = \{\eta_1, \ldots, \eta_l\}$ and let $\theta^i \subset \omega^{<\omega}$ be the sequence of siblings that contains η_i for $i = 1, \ldots, l$. Observe $\{\, \varphi(x, b_\mu) \mid \mu \in \theta^i \,\}$ is k-inconsistent for each $i = 1, \ldots l$. Because of the way we chose l, there is a path ν in the array $(b_{\theta^1} \ldots b_{\theta^l})$ such that $\{\, \varphi(x, b_{\nu(i)}) \mid i = 1, \ldots, l \,\}$ is inconsistent. By $\mathcal{L}_S$-indiscerniblity of $b_{\omega^{<\omega}}$, it holds that $b_{\nu(1)}, \ldots, b_{\nu(l)} \equiv b_{\eta_1} \ldots, b_{\eta_l}$, thus $\{\, \varphi(x, b_{\eta_i}) \mid i = 1, \ldots, l \,\}$ is inconsistent. $\boxed{\text{end of the proof of Claim}}$

By the Theorem 3.2.20, we have $\Gamma_{k'}$ is realized by an $\mathcal{L}_1$-indiscernible tree. □

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